On a Class of Strategy-proof Social Choice Correspondences with Single-peaked Utility Functions*

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Abstract

We consider the problem of constructing strategy-proof rules that choose sets of alternatives based on the preferences of voters, modelled as Social Choice Correspondences (SCCs) in the literature. We focus on two domain restrictions inspired by Barberà et al. (2001) in the context of single-peaked utility functions. We find that for the narrower domain, the set of tops-only, unanimous, and strategy-proof SCCs coincides with the class of unions of two min-max rules (Moulin (1980)). For the broader domain, the set of SCCs coincides with the class of unions of two 'adjacent' min-max rules, meaning the corresponding parameters for the two rules must be either the same or consecutive alternatives.

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1. Introduction

We consider the problem of choosing a set of alternatives based on the preferences of a group of agents over those alternatives. Such procedures are well known in the literature by the name of

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Social Choice Correspondences (SCCs). The outcome, as a set of alternatives, can be interpreted in many ways depending on the particular problem, such as collections of mutually compatible decisions as considered in Barberà et al. (1991) and Miyagawa (1997). In this paper, we aim to present the class of SCCs for two important sets of preferences of the agents (hereafter, called domains).

Our domain restrictions are motivated by those of Barberà et al. (2001). They consider preferences over subsets of alternatives that are extensions of a valuation function over individual alternatives. They study two types of extensions: first, *conditionally expected utility consistent* orderings (CEUC), and second, *conditionally expected utility consistent with equal probabilities orderings* (CEUCEP). In CEUC ordering, each agent has a utility function and a subjective probability distribution λ over the set of alternatives. The agent's valuation for any subset is the conditional expected utility of that set. Such a valuation over subsets can be interpreted in the following way: the outcome is a set of alternatives that are selected after a first round of screening, and the final alternative will be one from this set, chosen at a later stage.[1] The agents may not know the selection process in the second stage. As a result, they may have their subjective belief regarding this selection, which is captured by the probability distribution. Alternatively, another interpretation is that the outcome of the function becomes a decision-making committee, and λ is a voter's assessment of how much influence a given candidate has in making decisions. Hence a voter's implicit utility of a set of candidates S is the influence-weighted average over that set. In CEUCEP ordering, the only difference is that the underlying probability distribution λ is uniform over the set of alternatives (for another interpretation of CEUCEP ordering, see Barberà et al. (2001)). In both cases, they assume that the utility functions are unrestricted, i.e., a utility function maps to an element of $\mathbb{R}^m$, where m is the number of alternatives. Moreover, the utility function does not have a tie.

One of the most important classes of domains in social choice literature is the single-peaked domain. Single-peakedness requires an exogenous ordering over the set of alternatives. They appear naturally in a variety of situations, such as preference aggregation (Black (1948)), strategic voting (Moulin (1980)), public facility allocation (Bochet and Gordon (2012)), fair division (Sprumont (1991); Barberà et al. (1997)) and object assignment (Bade (2019)). Other than its wide applicability, single-peakedness is also important to circumvent the negative result of the seminal Gibbard-Satterthwaite Theorem (Gibbard (1973)-Satterthwaite (1975)). A utility function satisfies single-peakedness w.r.t. an ordering over the alternatives if there is some

[1]This interpretation is taken from Barberà et al. (2001).

alternative having the maximum utility, and as one moves away from it, the utility decreases.

In this paper, we assume that the agents have preferences over the subsets of alternatives that arise from a single-peaked utility function and a subjective probability distribution over the alternative set. As considered in the paper Barberà et al. (2001), we also work with two classes of domains, one where the probability distribution can be arbitrary (CESUC domain ($\mathcal{S}_U$)), whereas the other allows for only uniform probability distribution (CESUCEP domain ($\mathcal{S}_E$)). Given that the domain restriction considered in Barberà et al. (2001) is regarded as a meaningful way to derive preferences over the subset of alternatives by combining a utility function and a subjective belief over the alternative set, and the importance of single-peakedness, we believe that our domain restrictions are relevant.

Moulin (1980) shows that a social choice function (SCF) is tops-only, unanimous, and strategy-proof if and only if it is a min-max rule. A min-max rule is characterized by a set of parameters, one for each subset of the agents. At every preference profile, the outcome of the rule is determined by the peaks of the agents' preferences and the corresponding parameters. In the current work, we show that on the CESUC domain, an SCC is tops-only, unanimous, and strategy-proof if and only if it is a union of two min-max rules. We call such correspondences union min-max (UM) rules. This result implies that for such a correspondence, the outcome can be at most a doubleton set. For the CESUCEP domain, we show that the class of SCCs is more restrictive, a subclass of all union min-max rules. The SCCs on the CESUCEP domain are unions of two adjacent min-max rules where two min-max rules are adjacent if those have either the same or adjacent parameters for each subset. We name these correspondences as union adjacent min-max (UAM) rules.

One paper closely related to ours is Rodríguez-Álvarez (2017), which considers the single-peaked domain over the set of singleton and adjacent doubleton sets and shows that an SCF on this domain is unanimous and strategy-proof if and only if it is an extended median voter scheme (EMVS). Our domain restriction is more general than this as, in our case, the preferences are over all possible subsets of the alternatives. It is worth noting that if we restrict the CESUC domain over the singleton and adjacent doubleton sets, we get the domain restriction considered in Rodríguez-Álvarez (2017). Therefore, after showing that on the CESUC domain, any unanimous, strategy-proof, and tops-only SCC selects either a singleton or an adjacent doubleton set (Lemma 3), the characterization of corresponding SCCs follows from Rodríguez-Álvarez (2017). However, we prove this part independently of their result. Also, although any EMVS is a UAM rule and vice versa, the two definitions are structurally different. In Section 4.1, we comment on this in detail.

As discussed before, Barberà et al. (2001) consider two domain restrictions, the CEUC domain (henceforth $\mathcal{D}_E$) and the CEUCEP domain (henceforth $\mathcal{D}_U$). They characterize the unanimous and strategy-proof SCCs on these domains by showing that, on $\mathcal{D}_E$, an SCC is unanimous and strategy-proof if and only if it is a bi-dictatorial rule, whereas, on $\mathcal{D}_U$, an SCC is unanimous and strategy-proof if and only if it is a dictatorial rule. Our results are of similar flavour to theirs as we consider the corresponding two domains, $\mathcal{S}_E$ and $\mathcal{S}_U$, and characterize the tops-only, unanimous, and strategy-proof SCCs on those domains. Indeed, our results agree with theirs: dictatorial and bi-dictatorial rules are UM rules, and dictatorial rules are UAM rules.

The rest of the paper is organized as follows. Section 2 introduces the model and basic definitions. Section 3 presents our main result characterizing tops-only, unanimous, and strategy-proof SCCs on $\mathcal{S}_E$ and $\mathcal{S}_U$. Section 4 contains discussions of how our results are related to Rodríguez-Álvarez (2017) and provides an alternative characterization of tops-only, unanimous, and strategy-proof SCCs on $\mathcal{S}_U$. Section 5 gathers all omitted proofs.

2. Our Framework

Let $N = \{1, 2, \ldots, n\}$ be a set of at least two voters. Let $A = \{c_1, c_2, \ldots, c_m\}$ where $m \geq 3$ be a finite set of candidates with an intrinsic strict ordering $\prec$ given by $c_1 \prec c_2 \prec \cdots \prec c_m$. This can be interpreted as the stance of the candidates on some scale (eg. political leaning of candidates, ordered left to right). For $c_i, c_j \in A$, let $[c_i, c_j]$ denote the set of candidates between c_i and c_j, i.e., $[c_i, c_j] = \{c_i, c_{i+1}, \ldots, c_j\}$ if $c_i \prec c_j$ or $[c_i, c_j] = \{c_j, c_{j+1}, \ldots, c_i\}$ if $c_j \prec c_i$. Also, for convenience, we sometimes denote the candidate $c_{k\pm j}$ as $c_k \pm j$, for any j. Whenever we write minimum or maximum of a subset of A, we mean it with respect to the ordering $\prec$. By $a \preceq b$, we mean $a = b$ or $a \prec b$. Let $\mathcal{A}$ (henceforth, the alternative set) denote the nonempty subsets of A. We will use $\mathcal{A}_r$, $r = 1, \ldots, |A|$ to denote the set of subsets of A which have cardinality r. Throughout this paper, whenever it is clear from the context, we do not use braces for singleton sets.

2.1 Domain of Preferences

A preference R is a complete, reflexive, and transitive binary relation over the set $\mathcal{A}$.[2] For a preference P and for distinct $X, Y \in \mathcal{A}$, XRY is interpreted as "X is preferred to Y according to R". Further, the strict part of R is denoted by P and the indifference part of R is denoted by I.

[2] In general, whenever we write a preference over a subset B of $\mathcal{A}$, we mean a complete, reflexive, and transitive binary relation over the set B.

Let $\mathcal{R}$ be the set of all preferences over $\mathcal{A}$. An element of $\mathcal{R}^n$ is called a preference profile and is denoted by $R_N = (R_1, \ldots, R_n)$. For an agent $i \in N$, we use (R_i', R_{-i}) to denote the preference profile $(R_1, \ldots, R_{i-1}, R_i', R_{i+1}, \ldots, R_n)$.

We denote by $\mathcal{D} \subseteq \mathcal{R}$ a domain of admissible preferences. We will be interested in two types of domains in this paper which we describe below. We start with some definitions.

A **utility function** for individual i is a map $v_i : A \to \mathbb{R}$. Throughout the paper, we assume that for any voter i, and any distinct $x, y \in A$, $v_i(x) \neq v_i(y)$. A utility function v_i is **single-peaked** if there is a candidate $x \in A$ such that for all $y, z \in A$, $z \prec y \prec x \implies v_i(z) < v_i(y) < v_i(x)$ and $x \prec y \prec z \implies v_i(x) > v_i(y) > v_i(z)$. We call x the 'peak' of v_i. An **assessment** λ is a function $\lambda : A \to (0, 1]$ such that $\sum\limits_{c_i \in A} \lambda(c_i) = 1$.

We are now ready to describe the domains of interest in this paper. To ease the presentation, we introduce the following notation. For a utility function v_i and an assessment λ_i, we write

$$v_i^{\lambda_i}(X) = \sum_{x_j \in X} v_i(x_j) \left(\frac{\lambda_i(x_j)}{\sum\limits_{x_k \in X} \lambda_i(x_k)} \right) \text{ for all } X \in \mathcal{A}.$$

If λ_i is uniform, that is, $\lambda_i(x) = \frac{1}{m}$ for all $x \in A$, we will write $v_i^{\lambda_i}$ as v_i^E.

Definition 1. A preference R_i over $\mathcal{A}$ is said to be **conditionally expected single-peaked utility consistent (CESUC)** if there exists a single-peaked utility function v_i and an assessment λ_i such that:

$$X R_i Y \iff v_i^{\lambda_i}(X) \geq v_i^{\lambda_i}(Y) \text{ for all } X, Y \in \mathcal{A}.$$

Let $\mathcal{S}_U$ be the domain of all CESUC preferences.

Definition 2. A preference R_i over $\mathcal{A}$ is said to be **conditionally expected single-peaked utility consistent with equal probabilities (CESUCEP)** if there exists a single-peaked utility function v_i such that:

$$X R_i Y \iff v_i^E(X) \geq v_i^E(Y) \text{ for all } X, Y \in \mathcal{A}.$$

Let $\mathcal{S}_E$ be the domain of all CESUCEP preferences.

REMARK 1. Clearly, $\mathcal{S}_E \subset \mathcal{S}_U \subset \mathcal{R}$. □

We provide an example to illustrate the distinction between these two domains.

Example 1. Let $A = \{a, b, c\}$ with $a \prec b \prec c$. Consider the following preferences over $\mathcal{A}$:

1. The preference ordering R_1, with $\{a\} P_1 \{b\} P_1 \{a, b\} P_1 \{a, b, c\} P_1 \{a, c\} P_1 \{b, c\} P_1 \{c\}$. This preference R_1 belongs to $\mathcal{S}_U$ since it can be generated by the following single-peaked

utility function: $v(a) = 10, v(b) = 9, v(c) = 1$ and a uniform assessment λ_1, i.e., $\lambda_1(a) = \lambda_1(b) = \lambda_1(c) = \frac{1}{3}$. Since the preference can be generated by a uniform assessment, R_1 also belongs to $\mathcal{S}_E$.

2. The preference ordering R_2, with $\{a\}P_2\{b\}P_2\{a,c\}P_2\{a,b\}P_2\{a,b,c\}P_2\{b,c\}P_2\{c\}$. This preference R_1 belongs to $\mathcal{S}_U$ since it can be generated by the following single-peaked utility function: $v(a) = 10, v(b) = 2, v(c) = 1$ and the assessment λ_2 with $\lambda_2(a) = 0.1$, $\lambda_2(b) = 0.8, \lambda_2(c) = 0.1$. However, this preference cannot be generated by a uniform assessment, so R_2 is not a member of $\mathcal{S}_E$. Indeed, note that any preference R_j in $\mathcal{S}_E$ with $\tau(R_j) = a$ must have $\{a,b\}P_j\{a,c\}$.

These two examples illustrate how $\mathcal{S}_U$, by admitting arbitrary assessments, allows for a much richer domain of preferences than the more restrictive $\mathcal{S}_E$. Lemma 5, wherein we prove the membership of several families of preferences in $\mathcal{S}_E$ and $\mathcal{S}_U$, provides a deeper picture of the structure of these two domains.

Henceforth $\mathcal{S}$ denotes either $\mathcal{S}_U$ or $\mathcal{S}_E$, and we will only look at voter preferences with these domains. Note that $R_i \in \mathcal{S}$ implies that while R_i may exhibit indifference between some sets, the best and the worst alternatives according to R_i are singleton sets. The top-ranked alternative a of R_i is the peak of the underlying utility function v_i, whereas the worst alternative b of R_i is the alternative with the minimum utility in v_i. We denote this alternative a as $\tau(R_i)$.

For $R \in \mathcal{R}$ and $\mathcal{B} \subseteq \mathcal{A}$, the restriction of R to $\mathcal{B}$, $R|_\mathcal{B}$ is defined as follows: for all $X, Y \in \mathcal{B}$, $XR|_\mathcal{B}Y$ if and only if XRY. Similarly, we can define $P|_\mathcal{B}$ and $I|_\mathcal{B}$. For $\mathcal{B} \subseteq \mathcal{A}$, we define the restriction of $\mathcal{S}$ and $\mathcal{R}$ to $\mathcal{B}$ as $\mathcal{S}|_\mathcal{B} = \{R|_\mathcal{B} \mid R \in \mathcal{S}\}$ and $\mathcal{R}|_\mathcal{B} = \{R|_\mathcal{B} \mid R \in \mathcal{R}\}$ respectively. Further, the restriction of a profile $R_N \in \mathcal{R}^n$ to $\mathcal{B}$ is $R_N|_\mathcal{B} = (R_1|_\mathcal{B}, \dots, R_n|_\mathcal{B})$.

REMARK 2. Note that for $R \in \mathcal{R}$, $R|_{\mathcal{A}_1}$ is a preference over the candidates in A. Further, for $R \in \mathcal{S}$, $R|_{\mathcal{A}_1}$ has the following property: for any $a, b \in A$, either $a \prec b \prec \tau(R)$ or $\tau(R) \prec b \prec a$ implies $bP|_{\mathcal{A}_1}a$. Such a preference R is called a *single-peaked preference over* A, and $\mathcal{S}|_{\mathcal{A}_1}$ is the set of all single-peaked preferences over A. □

Later, when we discuss the connection between our paper and Rodríguez-Álvarez (2017), we require the following remark.

REMARK 3. Let $\mathcal{I} = \{S \subseteq A : S \text{ is an interval and } |S| \leq 2\}$, i.e., $\mathcal{I}$ is the set of all subsets of A that are either singletons or adjacent pairs. We can naturally extend the ordering $\prec$ over A to an ordering $\lhd$ on $\mathcal{I}$ in the following way: $\{c_1\} \lhd \{c_1, c_2\} \lhd \{c_2\} \lhd \{c_2, c_3\} \lhd \{c_3\} \lhd \cdots \lhd \{c_{m-1}, c_m\} \lhd \{c_m\}$. We write $x \unlhd y$ to denote $x \lhd y$ or $x = y$. A preference R on $\mathcal{I}$ is single-peaked with respect to the ordering $\lhd$ if there exists $x \in \mathcal{I}$ with $|x| = 1$ such that for all $y, z \in \mathcal{I}$, both $x \lhd y \lhd z$ and $z \lhd y \lhd x$ imply $xPyPz$. Let us denote $\mathcal{S}_{\mathcal{I}}$, the set of all single-peaked preferences on $\mathcal{I}$ with respect to $\lhd$. It is worth mentioning that for any preference $R \in \mathcal{S}$, $R|_{\mathcal{I}}$ is a single-peaked preference w.r.t. the ordering $\lhd$. Moreover, $\mathcal{S}|_{\mathcal{I}} = \mathcal{S}_{\mathcal{I}}$, i.e., if we restrict the preferences in $\mathcal{S}$ to $\mathcal{I}$, we get the set of all single-peaked preferences over $\mathcal{I}$. To see this fact, first, consider $R \in \mathcal{S}_{\mathcal{I}}$ and consider any single-peaked utility function v over A representing the same preferences over A as R, and the uniform assessment λ. Further, as for all $a, b \in A$, $v(a) < v(b)$ implies $v^E(a) < v^E(\{a, b\}) < v^E(b)$, we have the generated preference R' (using v and the uniform assessment) over $\mathcal{S}$ matches R over $\mathcal{I}$, so we conclude $\mathcal{S}_{\mathcal{I}} \subseteq \mathcal{S}_E|_{\mathcal{I}}$. To see the other inclusion, note that for any $R' \in \mathcal{S}|_{\mathcal{I}}$, the underlying utility function v under any assessment λ obeys that for $a, b \in A, v(a) < v(b) \implies v^{\lambda}(\{a, b\}) \in (v(a), v(b))$. In particular, this is true when a, b are adjacent. Along with the single-peakedness property of v, we have that R' is single-peaked over $\mathcal{I}$, hence $\mathcal{S}|_{\mathcal{I}} \subseteq \mathcal{S}_{\mathcal{I}}$. □

2.2 Social choice correspondences and their properties

In this section, we define social choice correspondences and discuss different properties of a social choice correspondence. Throughout the rest of the paper, we use $\mathcal{D} \subseteq \mathcal{R}|_{\mathcal{B}}$ to denote a domain with $\tau(R) \in \mathcal{A}_1$ for all $R \in \mathcal{D}$, where $\mathcal{B} \subseteq \mathcal{A}$.[3]

Definition 3. A **social choice correspondence** (SCC) f is a map $f : \mathcal{D}^n \to \mathcal{A}$.
An SCC $f : \mathcal{D}^n \to \mathcal{A}$ is a **social choice function** (SCF) if $|f(R_N)| = 1$ for all $R_N \in \mathcal{D}^n$. With a slight abuse of notation, we write $f : \mathcal{D}^n \to A$ to denote an SCF.

Definition 4. An SCC $f : \mathcal{D}^n \to \mathcal{A}$ is **tops-only** if for all $R_N, R'_N \in \mathcal{D}^n$

$$[\tau(R_i) = \tau(R'_i) \text{ for all } i \in N] \implies [f(R_N) = f(R'_N)].$$

The output of a tops-only SCC is only dependent on the most preferred candidate of each voter, rather than on their entire preference ordering. Hence, we can abuse notation to recast a tops-only SCC $f : \mathcal{D}^n \to \mathcal{A}$ as a function with domain A^n instead, i.e., we can write $f : A^n \to \mathcal{A}$. Such correspondences represent voting systems in which a voter can only indicate their most preferred candidate, and the outcome is chosen to be a subset of the candidates.

Definition 5. An SCC $f : \mathcal{D}^n \to \mathcal{A}$ is **manipulable** if there exist $R_N \in \mathcal{D}^n$, $i \in N$, and $R'_i \in \mathcal{D}$

[3]Note that the domains $\mathcal{S}$, $\mathcal{S}_{\mathcal{I}}$, and $\mathcal{S}|_{\mathcal{A}_1}$ satisfy this property and we will be using the definitions for these domains only.

such that $f(R_i', R_{-i}) P_i f(R_N)$. In such a situation, we say that agent i manipulates f at R_N via R_i'. If an SCC f is not manipulable, it is **strategy-proof**.

Definition 6. An SCC $f : \mathcal{D}^n \to \mathcal{A}$ is **unanimous** if for all $R_N \in \mathcal{D}^n$,

$$[\tau(R_1) = \cdots = \tau(R_n)] \implies [f(R_N) = \tau(R_1)].$$

Recall that for any preference R in $\mathcal{S}$, $\tau(R) = a$ for some $a \in A$. Thus, for any unanimous SCC on $\mathcal{S}$, a singleton set will be the outcome in any unanimous profile.

2.2.1 A class of social choice correspondences

In this paper, we introduce two new classes of SCCs. These SCCs are defined using min-max rules introduced by Moulin (1980). We first recall the definition of a min-max rule.

Definition 7. An SCF $f : \mathcal{D}^n \to A$ is a **min-max** rule if there exists $\{\beta_S^f\}_{S \subseteq N}$ with $\beta_S^f \in A$ for every $S \subseteq N$ satisfying the following two conditions

(i) $\beta_\emptyset^f = c_m$, $\beta_N^f = c_1$, and

(ii) $\beta_T^f \preceq \beta_S^f$ for $S \subseteq T$,

such that for all $R_N \in \mathcal{D}^n$

$$f(R_N) = \min_{S \subseteq N}[\max_{i \in S}\{\tau(R_i), \beta_S^f\}].$$

Note that a min-max rule is tops-only by definition. Therefore, we sometimes use a slight abuse of notation to write a min-max rule as a function from A^n to A. We now define the two new classes of SCCs.

Definition 8. An SCC $f : \mathcal{D}^n \to \mathcal{A}$ is a **union min-max** (UM) rule if there exist two min-max rules $g, h : \mathcal{D}^n \to A$ such that for all $R_N \in \mathcal{D}^n$

$$f(R_N) = g(R_N) \cup h(R_N).$$

Two min-max rules $g, h : \mathcal{D}^n \to A$ are called **adjacent min-max** rules if $\left|[\beta_S^g, \beta_S^h]\right| \leq 2$ for all $S \subseteq N$.

Definition 9. An SCC $f : \mathcal{D}^n \to \mathcal{A}$ is a **union adjacent min-max** (UAM) rule if there exist two adjacent min-max rules $g, h : \mathcal{D}^n \to A$ such that for all $R_N \in \mathcal{D}^n$

$$f(R_N) = g(R_N) \cup h(R_N).$$

A couple of facts follow immediately from the above definitions. The first is that any UAM rule is also a UM rule. The second is that any UM rule chooses an element of $\mathcal{A}_1 \cup \mathcal{A}_2$ at any preference profile. In what follows, we provide two examples, one for a UM rule and another for a UAM rule.

Example 2. Let $A = \{a, b, c\}$ with $a \prec b \prec c$ and $N = \{1, 2\}$. Consider the following SCC $f : \mathcal{S}^2 \to \mathcal{A}$.

$$f(R_1, R_2) = \min\{\tau(R_1), \tau(R_2)\} \cup \max\{\tau(R_1), \tau(R_2)\}.$$

We claim that f is a UM rule. To see this, first note that $g : \mathcal{S}^2 \to A$ defined as

$$g(R_1, R_2) = \min\{\tau(R_1), \tau(R_2)\}.$$

is a min-max rule. This can be verified with the choices of $\beta_1^g = \beta_2^g = a$. Similarly, we can show that $h : \mathcal{S}^2 \to A$ defined as

$$h(R_1, R_2) = \max\{\tau(R_1), \tau(R_2)\}.$$

is a min-max rule with $\beta_1^h = \beta_2^h = c$. Hence, $f(R_1, R_2) = g(R_1, R_2) \cup h(R_1, R_2)$ is a UM rule. However, as g and h are not adjacent min-max rules, f is not a UAM rule. □

Example 3. Let $A = \{a, b, c\}$ with $a \prec b \prec c$ and $N = \{1, 2\}$. Consider the following SCC $f : \mathcal{S}^2 \to \mathcal{A}$.

$$f(R_1, R_2) = \begin{cases} \max\{\tau(R_1), \tau(R_2)\} & \text{if } \max\{\tau(R_1), \tau(R_2)\} \prec c, \\ \{b, c\} & \text{if } \max\{\tau(R_1), \tau(R_2)\} = c \text{ and } \min\{\tau(R_1), \tau(R_2)\} \prec c, \\ c & \text{otherwise.} \end{cases}$$

We assert that the SCC f is a UAM rule. Consider the following two SCFs g and h

$$g(R_1, R_2) = \begin{cases} \max\{\tau(R_1), \tau(R_2)\} & \text{if } \max\{\tau(R_1), \tau(R_2)\} \prec c, \\ b & \text{if } \max\{\tau(R_1), \tau(R_2)\} = c \text{ and } \min\{\tau(R_1), \tau(R_2)\} \prec c, \\ c & \text{otherwise,} \end{cases}$$

$$h(R_1, R_2) = \max\{\tau(R_1), \tau(R_2)\}$$

It is clear that $f(R_1, R_2) = g(R_1, R_2) \cup h(R_1, R_2)$ for all $(R_1, R_2) \in \mathcal{S}^2$. Also, it can be checked that g and h are min-max rules given by the parameters $\beta_1^g = \beta_2^g = b$ and $\beta_1^h = \beta_2^h = c$. Thus, g and h are adjacent min-max rules and hence, f is a UAM rule.

One crucial observation about f is that the outcome of f at every profile is an interval. Later, we show that, in general, this is true for any UAM rule (see Claim 1). □

3. Characterization Results

We first state a known result in the context of SCFs (Moulin (1980), Weymark (2011)). Recall that $\mathcal{S}|_{\mathcal{A}_1}$ is the set of all single-peaked preferences over A.

Theorem (Moulin (1980), Weymark (2011))**.** *An SCF $f : (\mathcal{S}|_{\mathcal{A}_1})^n \to A$ is tops-only, unanimous, and strategy-proof if and only if it is a min-max rule.*

It is straightforward to see that if we consider tops-only, unanimous, and strategy-proof SCFs on $\mathcal{S}$, the same result holds. Therefore, we have the following fact. For completeness, we provide proof of this in the Appendix 5.1.

Fact 1. *An SCF $f : \mathcal{S}^n \to A$ is tops-only, unanimous, and strategy-proof if and only if it is a min-max rule.*

We now present the two main results in this paper. Theorem 1 and Theorem 2 characterize the set of tops-only, unanimous, and strategy-proof SCCs on the domains $\mathcal{S}_E$ and $\mathcal{S}_U$, respectively.

Theorem 1. *Let $f : \mathcal{S}_E^n \to \mathcal{A}$. Then f is tops-only, unanimous, and strategy-proof if and only if it is a UM rule.*

Theorem 2. *Let $f : \mathcal{S}_U^n \to \mathcal{A}$. Then f is tops-only, unanimous, and strategy-proof if and only if it is a UAM rule.*

3.1 Proof of Theorem 1

Proof of the "if" part: We show that any UM rule is tops-only, unanimous, and strategy-proof on $\mathcal{S}_E$. Let g, h be two min-max rules and f the UM rule such that $f(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{S}_E^n$. Since min-max rules are tops-only and unanimous, f is also tops-only and unanimous. We show that f is strategy-proof. As f, g, and h tops-only, we may write these as functions with domain A^n, that is, $f : A^n \to \mathcal{A}$ and $g, h : A^n \to A$. Assume for contradiction that f is not strategy-proof. WLOG assume that voter 1 can manipulate at a profile R_N via a preference R_1'. Let's denote $a_j = \tau(R_j)$ for all $j \in N$ and $a_1' = \tau(R_1')$. Since $R_1 \in \mathcal{S}_E$, there exists a utility function v_1, generating R_1. Thus, we have $v_1^E(f(a_1,\ldots,a_n)) < v_1^E(f(a_1',\ldots,a_n))$ which further implies

$$v_1^E(\{g(a_1,\ldots,a_n)\}\cup\{h(a_1,\ldots,a_n)\}) < v_1^E(\{g(a_1',\ldots,a_n)\}\cup\{h(a_1',\ldots,a_n)\})$$
$$\implies \frac{v_1(g(a_1,\ldots,a_n))+v_1(h(a_1,\ldots,a_n))}{2} < \frac{v_1(g(a_1',\ldots,a_n))+v_1(h(a_1',\ldots,a_n))}{2}.$$

This above inequality requires either $v_1(g(a_1',\ldots,a_n)) > v_1(g(a_1,\ldots,a_n))$ or $v_1(h(a_1',\ldots,a_n)) > v_1(h(a_1,\ldots,a_n))$. This is a contradiction since both g and h are min-max rules, and hence by Fact 1, these are strategy-proof on $\mathcal{S}$. ■

Proof of the "only-if" part: Let $f:\mathcal{S}_E^n\to\mathcal{A}$ be a tops-only, unanimous, and strategy-proof SCC. We will show that f is a UM rule, that is, there exist min-max rules g and h such that $f(R_N)=g(R_N)\cup h(R_N)$ for all $R_N\in\mathcal{S}_E^n$. We first prove three propositions unfolding different properties of f. The first proposition shows that f satisfies uncompromisingness. For an SCF, uncompromisingness was first introduced in Border and Jordan (1983). We slightly modify the definition to fit it in the context of an SCC. Roughly speaking, an SCC is uncompromising if, for any candidate x, a voter changing their peak while remaining strictly on the same side of x as before does not change the membership of x in the outcome. Below, we formally define uncompromisingness of an SCC.

Definition 10. An SCC $f:\mathcal{S}_E^n\to\mathcal{A}$ is **uncompromising** if for all $R_N\in\mathcal{S}_E^n$, all $R_i'\in\mathcal{S}_E$, and all $x\in A\setminus[\tau(R_i),\tau(R_i')]$, we have

$$x\in f(R_N)\iff x\in f(R_i',R_{-i}).$$

We now state the first proposition.

Proposition 1. *f is uncompromising.*

Proof. We divide the proof into two lemmas, first one showing that f satisfies the interval property, that is, for any profile $R_N\in\mathcal{S}_E^n$, the outcome of f is a subset of the interval

$$[\min\{\tau(R_1),\ldots,\tau(R_n)\},\max\{\tau(R_1),\ldots,\tau(R_n)\}]$$

Lemma 1. *For all $R_N \in \mathcal{S}_E^n$,*

$$f(R_N) \subseteq [\min\{\tau(R_1), \ldots, \tau(R_n)\}, \max\{\tau(R_1), \ldots, \tau(R_n)\}] .$$

Proof. Since f is tops-only, we can recast it as a function $f : A^n \to \mathcal{A}$. Suppose there exists $x \in f(x_1, \ldots, x_n), x \notin [\min\{x_1, \ldots, x_n\}, \max\{x_1, \ldots, x_n\}]$. WLOG say $x \prec \min\{x_1, \ldots, x_n\}$. Then consider the following sets:

$$\begin{aligned} A_1 &= f(x_1, x_2, x_3, x_4, \ldots, x_n) \\ A_2 &= f(x_1, x_1, x_3, x_4, \ldots, x_n) \\ A_3 &= f(x_1, x_1, x_1, x_4, \ldots, x_n) \\ &\vdots \\ A_n &= f(x_1, x_1, x_1, x_1, \ldots, x_1) \end{aligned}$$

Note that $x \in A_1$, but $A_n = \{x_1\}$ (by unanimity). Consider the smallest i such that $[c_1, x] \cap A_i = \emptyset$. Let voter i have a preference ordering R_i that peaks at x_i such that $Y P_i X$ whenever $[c_1, x] \cap Y = \emptyset, [c_1, x] \cap X \neq \emptyset$ (see (i) of Lemma 5 in Appendix 5.2). Then, i can manipulate at $f(x_1, \ldots, x_1, x_i, x_{i+1}, \ldots, x_n)$ to $f(x_1, \ldots, x_1, x_1, x_{i+1}, \ldots, x_n)$ for a better outcome, since the first contains an element of $[c_1, x]$ while the second does not. Since f is strategy-proof, this is a contradiction. ■

The second lemma exhibits another crucial property of f. It shows that the cardinality of f is always at most two.

Lemma 2. *For all $R_N \in \mathcal{S}_E^n$, $|f(R_N)| \leq 2$.*

We relegate the proof of this lemma in Appendix 5.3. Given Lemma 1 and 2, we are ready to prove Proposition 1.

Since f is tops-only, we recast it as a function $f : A^n \to \mathcal{A}$. Assume otherwise, that the proposition does not hold for some voter i and for some profile $(a_1, \ldots, a_i, \ldots, a_n)$, i.e., there exists some $x \notin \{a_i, a_i + 1\}$ whose membership in the outcome set is affected by voter i switching their peak from a_i to $a_i + 1$. We may assume that $x \succ a_i + 1$; if $x \prec a_i$, we can simply look at the reverse ordering over candidates. Further, take x to be the be the largest candidate whose membership is affected by this switch. Define $A := f(a_1, \ldots, a_i, \ldots, a_n), B := f(a_1, \ldots, a_i + 1, \ldots, a_n)$.

Case 1: $x = \max(A \cup B)$

Let voter i have a preference R_i such that XP_iY whenever $X \cap [x, c_m] = \emptyset$ and $Y \cap [x, c_m] \neq \emptyset$ (there exist such preferences that peak at each of a_i and $a_i + 1$; (see (iii) of Lemma 5 in Appendix 5.2). Then i can choose between voting between voting for a_i and $a_i + 1$ depending on whichever outcome set excludes x for a better outcome, regardless of their own whether their preference peaks at a_i or $a_i + 1$. Hence strategy-proofness is broken.

Case 2: $x \neq \max(A \cup B)$. So there exists $y \succ x$ with $y \in A \cap B$.

(i) If either $A = \{y\}$ or $B = \{y\}$: Let i have a preference R'_i such that XP'_iY whenever $X \nsubseteq [x+1, c_m], Y \subseteq [x+1, c_m]$ (there exist such preferences that peak at each of a_i and $a_i + 1$; (see (iv) of Lemma 5 in Appendix 5.2). Then i can choose to vote for whichever of $a_i, a_i + 1$ would cause the outcome set to include x regardless of their own preference. Since $|A| \leq 2, |B| \leq 2$ (from Lemma 2), this set must be exactly $\{x, y\}$, which is preferred over $\{y\}$. They do this regardless of whether their preference peaks at a_i or $a_i + 1$. Hence, strategy-proofness is broken.

(ii) If $A \neq \{y\}$ and $B \neq \{y\}$: Then one of A, B is of the form $\{x, y\}$ and the other is of the form $\{w, y\}$ for some $w \prec x \prec y$. Then let i have the preference R''_i such that XP''_iY whenever $X \nsubseteq [x, c_m], Y \subseteq [x, c_m]$ (there exist such utilities that peak at each of a_i and $a_i + 1$; (see (iv) of Lemma 5 in Appendix 5.2). Then i can choose the set that excludes x for a better outcome regardless of whether their peak is at a_i or $a_i + 1$. Hence, the function is not strategy-proof.

In all cases, we get that f is not strategy-proof, yielding contradictions. Hence, f must be uncompromising. ■

From uncompromisingness, we know when an agent unilaterally moves her peak from one candidate to the next one, the membership of any candidate other than the two peaks remains unchanged. However, uncompromisingness does not say anything about the membership of the two adjacent peaks. The following proposition directs us to how they behave in such a scenario.

Proposition 2. *Let $A := f(R_N)$ and $B := f(R'_i, R_{-i})$ where $\tau(R_i)$ and $\tau(R'_i)$ are adjacent and $\tau(R_i) \prec \tau(R_{i+1})$ for some $R_N \in \mathcal{S}_E^n$, $i \in N$, and $R'_i \in \mathcal{S}_E$. Define $S := A \cap \{\tau(R_i), \tau(R'_i)\}$ and $T := B \cap \{\tau(R_i), \tau(R'_i)\}$. Then:*

- *if $S = \emptyset$ then $T = \emptyset$,*

- *if* $S = \{\tau(R_i)\}$ *then* $T = \{\tau(R_i)\}$ *or* $\{\tau(R_i), \tau(R'_i)\}$ *or* $\{\tau(R'_i)\}$,
- *if* $S = \{\tau(R_i), \tau(R'_i)\}$ *then* $T = \{\tau(R_i), \tau(R'_i)\}$ *or* $\{\tau(R'_i)\}$, *and*
- *if* $S = \{\tau(R'_i)\}$ *then* $T = \{\tau(R'_i)\}$.

Proof. Consider a preference profile $R_N \in \mathcal{S}_E^n$, $i \in N$, and $R'_i \in \mathcal{S}_E$. To facilitated the writing, let's denote $\tau(R_i)$ by a_i and $\tau(R'_i)$ by $a_i + 1$. Note that the rest of the outcomes besides a_i and $a_i + 1$ are unchanged between A and B due to uncompromisingness.

If $S = \emptyset$ and $T \neq \emptyset$ then f is manipulated by i when i has a preference R_i such that $\forall a \in A \setminus \{a_i, a_i + 1\}$, $\{a_i\} P_i \{a_i + 1\} P_i \{a\}$. Then i votes $a_i + 1$ for a better outcome even though they prefer a_i.

If $S = \{a_i\}$ and $T = \emptyset$ then f is manipulated by i when i has a preference R_i such that $\forall a \in A \setminus \{a_i, a_i + 1\}$, $\{a_i + 1\} P_i \{a_i\} P_i \{a\}$, since i can vote a_i instead of $a_i + 1$ for a better outcome.

If $S = \{a_i, a_i + 1\}$ and $T = \{a_i\}$, then f is manipulated by i when i has a preference R'_i that peaks at a_i, since i can vote $a_i + 1$ for a better outcome.

If $S = \{a_i + 1\}$ and $a_i \in T$, then f is manipulated by i when i has a preference R''_i that peaks at a_i. If $S = \{a_i + 1\}$ and $T = \emptyset$, then f is manipulated by i when i has a preference R'''_i that peaks at $a_i + 1$, since i can vote a_i instead of $a_i + 1$ for a better outcome. ∎

REMARK 4. An analogous result holds for the leftward movement of voter i's peak.

The following proposition shows that if two SCCs agree on all the profiles where agents have either c_1 or c_m as the peak, then those two SCCs will yield the same outcome everywhere on $\mathcal{S}_E^n$. A similar result is shown by Peters et al. (2014) in the context of probabilistic rules.

Proposition 3. *Let* $f_1, f_2 : \mathcal{S}_E^n \to \mathcal{A}$ *be tops-only, unanimous, and strategy-proof SCCs. If* $f_1(R_N) = f_2(R_N)$ *when* $\tau(R_i) \in \{c_1, c_m\}$ *for all* $i \in N$, *then* $f_1(R_N) = f_2(R_N)$ *for all* $R_N \in \mathcal{S}_E^n$.

Proof. As usual, since f_1, f_2 are tops-only, we recast them to be functions $f_1, f_2 : A^n \to \mathcal{A}$. We prove the proposition by contradiction. Suppose we have that $f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n)$ when $a_i \in \{c_1, c_m\}$ for all $i \in N$, but that for some particular $(x_1, \ldots, x_n) \in A^n$ there is a candidate c_j that belongs to only one of $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$. Let $S = \{i : x_i \prec c_j\}$, $T = \{i : x_i \succ c_j\}$. Define

$$B(x_i) = \begin{cases} c_1 & i \in S \\ c_m & i \in T \\ x & \text{otherwise} \end{cases}$$

From Proposition 2, we note that $c_j \in f_1(x_1, \ldots, x_n) \iff c_j \in f_1(B(x_1), \ldots, B(x_n))$.

Case 1: Suppose $c_j \neq x_i$ for any i. Then,

$$\begin{aligned} c_j \in f_1(x_1, \ldots, x_n) &\iff c_j \in f_1(B(x_1), \ldots, B(x_n)) \\ &\iff c_j \in f_2(B(x_1), \ldots, B(x_n)) \\ &\iff c_j \in f_2(x_1, \ldots, x_n) \end{aligned}$$

We have a contradiction.

Case 2: Alternatively, suppose $c_j = x_{i_1} = \ldots = x_{i_n}$ for some $i_1, \ldots, i_n$. Let $\overline{f_1}, \underline{f_1}$ be the maximum and minimum of $f_1(x_1, \ldots, x_n)$, and $\overline{f_2}, \underline{f_2}$ be the maximum and minimum of $f_2(x_1, \ldots, x_n)$. If $\overline{f_1} = \overline{f_2}$ and $\underline{f_1} = \underline{f_2}$, then it must be the case that $f_1(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n)$ (since by Lemma 2 each of these sets have at most two elements). This is a contradiction, hence one of $\overline{f_1} = \overline{f_2}$ and $\underline{f_1} = \underline{f_2}$ is not true. WLOG say $\overline{f_1} \neq \overline{f_2}$, and further WLOG say $\overline{f_1} \succ \overline{f_2}$, and that $c_j = \overline{f_1}$. We note that $x_{i_1} = \ldots = x_{i_n} = c_j = \overline{f_1}$. Define

$$B'(x_i) = \begin{cases} c_1 & i \in S \\ c_m & i \in T \\ c_m & \text{otherwise} \end{cases}$$

But now, we have

$$\begin{aligned} [c_j, c_m] \cap f_1(B'(x_1), \ldots, B'(x_n)) \neq \varnothing \\ [c_j, c_m] \cap f_2(B'(x_1), \ldots, B'(x_n)) = \varnothing \end{aligned}$$

Repeated application of Proposition 2 gives us the first fact, and Lemma 1 gives us the second. We get the contradiction that our functions do not agree even on boundary inputs. Hence, there is no such candidate c_j. ■

We are now ready to complete the proof of Theorem 1. From Lemma 2, we know that $|f(R_N)| \leq 2$. Define $g(R_N) := \min f(R_N)$ and $h(R_N) := \max f(R_N)$. We simply need to show that g and h are min-max rules. We shall prove this for h, the proof for g is analogous. In particular, we shall prove that h as defined here is tops-only, unanimous, and strategy-proof SCF on $\mathcal{S}_E$, and the proof would follow by Fact 1.

Since f is tops-only, it must be that h is tops-only. Hence we recast f, h to have domain A^n. It is clear that h is unanimous from the unanimity of f. Suppose h is not strategy-proof; suppose it is manipulable by (WLOG) voter 1. Let $(a_1, \ldots, a_n) \in A^n$ be a profile and $a'_1 \neq a_1$ be a candidate such that when voter 1 has a single-peaked utility v_1 that peaks at a_1, they can manipulate at $(a_1, \ldots, a_n)$ to get a more preferred outcome $h(a'_1, \ldots, a_n)$.

Suppose $a_1 \prec a'_1$. Then, we must have that $h(a_1, \ldots, a_n) \in [a_1, a'_1]$, otherwise due to the uncompromisingness of f and therefore h (Lemma 1), the outcome would simply not move. However, if $h(a_1, \ldots, a_n) \in [a_1, a'_1]$, Proposition 2 prescribes how the outcome set can move in response to voter 1 moving rightwards: elements of the outcome of f can only move rightwards, and hence the outcome of h can only move rightwards. Hence we get that $h(a'_1, \ldots, a_n) \succ h(a_1, \ldots, a_n)$. This is a contradiction, since the new outcome is further from the peak and is in fact less preferred, not more. Similar reasoning holds to get a contradiction for if $a_1 \succ a'_1$.

Hence, h is strategy-proof. In a similar manner, g can be shown to be unanimous and strategy-proof. From Fact 1, we know that g, h are therefore min-max rules. Hence we have that $\forall (x_1, \ldots, x_n) \in A^n$, $f(R_N) = \{g(R_N)\} \cup \{h(R_N)\}$ for min-max rules g, h. This completes the proof of the only-if part of the theorem. ■

3.2 Proof of Theorem 2

Proof of the "if" part: We show that any UAM rule is tops-only, unanimous, and strategy-proof on $\mathcal{S}_U$. Let g, h be two adjacent min-max rules and f the UAM rule such that $f(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{S}_U^n$. Since min-max rules are tops-only and unanimous, f is also tops-only and unanimous. We show that f is strategy-proof. Recall the set $\mathcal{I} = \{S \subseteq A : S \text{ is an interval and } |S| \leq 2\}$ and the ordering $\lhd$ defined in Remark 3. We first prove a claim.

Claim 1. *$f(R_N) \in \mathcal{I}$ for all $R_N \in \mathcal{S}_U^n$.*

Proof of Claim 1: Recall that by our assumption $f(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{S}_U^n$ where g, h are two adjacent min-max rules. As f, g, and h are tops-only functions where the tops are elements of A, we write them as functions from A^n to A. Assume for contradiction there exists $(a_1, \ldots, a_n) \in A^n$, such that $f(a_1, \ldots, a_n) \notin \mathcal{I}$. Let $x, y \in f(a_1, \ldots, a_n)$ with $x \prec y - 1$. WLOG we may assume that $a_1, \ldots, a_{j-1} \preceq x$ and $a_j, \ldots, a_n \succ x$ for some $j \leq n$. Also, as $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n) \cup h(a_1, \ldots, a_n)$, we may assume that $g(a_1, \ldots, a_n) = x$ and $h(a_1, \ldots, a_n) = y$. Consider the profile $\boldsymbol{c} = (\underbrace{c_1, \ldots, c_1}_{j-1}, \underbrace{c_m, \ldots, c_m}_{n-j+1})$, that is obtained by moving $x_1, \ldots, x_{j-1}$ to c_1 and moving $x_j, \ldots, x_n$ to c_m.

Since g and h are min-max rules, we must have $g(\boldsymbol{c}) \prec x$ and $h(\boldsymbol{c}) \succ y$. This means $\beta_S^g \prec x$

and $\beta^h_S \succ y$ where $S = \{1, \ldots, j-1\}$. But this contradicts the fact that g and h are adjacent min-max rules. $\square$

We now proceed to prove that f is strategy-proof. Since f is tops-only, we recast f as a function $f : A^n \to \mathcal{A}$. Note that if an agent j has a preference ordering $R_j \in \mathcal{S}_U$, their preferences over elements of $\mathcal{I}$ must be single-peaked with respect to the ordering $\lhd$, with the same peak $\tau(R_j)$. Consider a profile $(a_1, \ldots, a_n) \in A^n$, an agent $i \in N$, and a peak $a'_i \in A$. We show that f is not manipulable by agent i at $(a_1, \ldots, a_n)$ via a'_i. WLOG assume that $a'_i \succ a_i$.

Case 1: $f(a_1, \ldots, a_n) \lhd a_i$

Since $f(a_1, \ldots, a_n) \lhd a_i \lhd a'_i$, we have $g(a_1, \ldots, a_n) \preceq a_i \preceq a'_i$ and $h(a_1, \ldots, a_n) \preceq a_i \preceq a'_i$. Thus, the outcome of g and h do not change from $(a_1, \ldots, a_i, \ldots, a_n)$ to $(a_1, \ldots, a'_i, \ldots, a_n)$ and so neither does the outcome of f. Hence, the agent i cannot manipulate f.

Case 2: $f(a_1, \ldots, a_n) = a_i$.

Since i already has the best possible outcome according to R_i, i cannot manipulate f.

Case 3: $f(a_1, \ldots, a_n) \rhd a_i$.

Note that by the definition of f, $f(a_1, \ldots, a'_i, \ldots, a_n) \in \mathcal{I}$. Further, as $a'_i \succ a_i$ and g and h are min-max rules, it follows that $g(a_1, \ldots, a'_i, \ldots, a_n) \succeq g(a_1, \ldots, a_i, \ldots, a_n)$ and $h(a_1, \ldots, a'_i, \ldots, a_n) \succeq h(a_1, \ldots, a_i, \ldots, a_n)$. Combining the two observations with the assumption of the case, we have $f(a_1, \ldots, a'_i, \ldots, a_n) \rhd f(a_1, \ldots, a_i, \ldots, a_n) \rhd a_i$. However, as for any $R \in \mathcal{S}_U$, $R|_{\mathcal{I}}$ is single-peaked with the peak at $\tau(R_i)$, we have for any preference of agent i with a_i at the peak, $f(a_1, \ldots, a_i, \ldots, a_n)$ is preferred over $f(a_1, \ldots, a'_i, \ldots, a_n)$. Hence, agent i cannot manipulate f.

As the Cases 1, 2, and 3 are exhaustive, we have f is strategy-proof. This completes the proof of the if part. $\blacksquare$

Proof of the "only-if" part: Let $f : \mathcal{S}^n_U \to \mathcal{A}$ be a tops-only, unanimous, and strategy-proof SCC. We will show that f is a UAM rule, that is, there exist two adjacent min-max rules g and h such that $f(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{S}^n_U$. Recall from Remark 1 that $\mathcal{S}_E \subseteq \mathcal{S}_U$. Consider the restriction of f to $\mathcal{S}^n_E$, and let's denote it by f'. Since $\mathcal{S}_E \subseteq \mathcal{S}_U$, f' is tops-only, unanimous, and strategy-proof on $\mathcal{S}_E$. Thus, by Theorem 1, f' is a UM rule, and by the definition of a UM rule there exist two min-max rules g and h such that $f'(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{S}^n_E$. Since f is tops-only, this means $f(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{S}^n_U$. We prove the theorem by showing that g and h are two adjacent min-max rules, i.e., $\left|[\beta^g_S, \beta^h_S]\right| \leq 2$ for all $S \subseteq N$. In the following, we show a crucial property of f.

Lemma 3. *For all $R_N \in \mathcal{S}^n_U$, $f(R_N) \in \mathcal{I}$.*

Proof. Since f is tops-only, we can recast it as a function $f : A^n \to \mathcal{A}$. Assume for contradiction that there exist $(x_1, \ldots, x_n) \in A^n$ and $b \in A$ such that $b \notin [\min f(x_1, \ldots, x_n), \max f(x_1, \ldots, x_n)]$. Let us write $a = \min f(x_1, \ldots, x_n)$ and $c = \max f(x_1, \ldots, x_n)$. Also, without loss of generality, we may assume that $x_1, \ldots, x_{i-1} \prec c$ and $x_i, \ldots, x_n \succeq c$ for some $i \leq n$.

Consider the outcome $f(x_1, \ldots, x_{i-1}, c, \ldots, c)$. As f is uncompromising and $x_i, \ldots, x_n \succeq c$, for any $x \prec c$,

$$x \in f(x_1, \ldots, x_n) \iff x \in f(x_1, \ldots, x_{i-1}, c, \ldots, c).$$

Moreover, as $c \in f(x_1, \ldots, x_n)$ and $x_i, \ldots, x_n \succeq c$, by uncompromisingness of f and Proposition 2, $c \in f(x_1, \ldots, x_{i-1}, c, \ldots, c)$. Finally, for any $x \succ c$, as $x \notin f(x_1, \ldots, x_n)$, by uncompromisingness of f, $x \notin f(x_1, \ldots, x_{i-1}, c, \ldots, c)$. Combining all the observations, we have $f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, c, \ldots, c)$. Define:

$$\begin{aligned} Y_{i-1} &:= f(x_1, \ldots, x_{i-1}, c, c, c, \ldots, c) = f(x_1, \ldots, x_n) \\ Y_i &:= f(x_1, \ldots, x_{i-1}, b, c, c, \ldots, c) \\ Y_{i+1} &:= f(x_1, \ldots, x_{i-1}, b, b, c, \ldots, c) \\ &\vdots \\ Y_n &:= f(x_1, \ldots, x_{i-1}, b, b, b, \ldots, b) \end{aligned}$$

We make the following observations. First, for any $k \geq i$, $\max Y_k \succeq b$. To see this, suppose $l \geq i$ is the first index such that $\max Y_l \prec b$. Then, by uncompromisingness of f, $\max Y_{l-1} \prec b$. If $l = i$ then this contradicts that $\max Y_{i-1} = c \succ b$, otherwise this contradicts the fact that l is the first index with $\max Y_l \prec b$. Hence, for any $k \geq i$, $\max Y_k \succeq b$. Second, $\max Y_n = b$, otherwise, if $\max Y_n \succ b$, by moving $x_1, \ldots, x_{i-1}$, one by one, to b and applying uncompromisingness of f, we would get $\max f(b, \ldots, b) \succ b$, contradicting unanimity of f; hence $\max Y_n = b$. Therefore, there exists $j \geq i$ such that $\max Y_j = b$ and $\max Y_{j-1} \succ b$. Without loss of generality, we may assume that j is first index with such property. Moreover, as $a \in Y_{j-1}$ and $\max Y_{j-1} \succ b$, $|Y_{j-1}| \leq 2$ implies $b \notin Y_{j-1}$.

Let R_j be a preference with peak at b and $Y_{j-1} P_j Y_j$ (see (v) of Lemma 5 in Appendix 5.2). Note that such a preference exists as $b \notin Y_{j-1}$ and $\max Y_j = b \prec \max Y_{j-1}$. Then j can manipulate at $(x_1, \ldots, x_{i-1}, \underbrace{b, \ldots, b}_{j-i+1}, c, \ldots, c)$ via c. This contradicts the fact that f is strategy-proof. ■

We now complete the proof of the only-if part. Note that as $f(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{S}_U^n$ and $f(R_N)$ is an interval for all $R_N \in \mathcal{S}_U^n$ (Lemma 3), we have $|[g(R_N), h(R_N)]| \leq 2$ for all $R_N \in \mathcal{S}_U^n$. Consider $S \subseteq N$ and a preference profile $\bar{R}_N$ in $\mathcal{S}_U^n$ where $\tau(\bar{R}_i) = c_1$ for all $i \in S$ and $\tau(\bar{R}_i) = c_m$ for all $i \in N \setminus S$. Note that by the definition of a min-max rule, $g(\bar{R}_N) = \beta_S^g$ and $h(\bar{R}_N) = \beta_S^h$. Therefore, $|[g(\bar{R}_N), h(\bar{R}_N)]| \leq 2$ implies $\left|[\beta_S^g, \beta_S^h]\right| \leq 2$. This completes the proof of the only-if part. ■

4. Further Discussions

4.1 An alternative characterization of SCCs over $\mathcal{S}_U$

As discussed in Section 1, Rodríguez-Álvarez (2017) works with a similar framework as ours. The author assumes single-peaked preferences on $\mathcal{I}$ with respect to the ordering $\lhd$.[4] Rodríguez-Álvarez (2017) proves that the set of unanimous and strategy-proof SCCs on $\mathcal{S}_\mathcal{I}$ is exactly the set of all *extended median voter schemes* (EMVSs) (see Theorem 3 in Rodríguez-Álvarez (2017)). Hereafter, we provide a formal definition of an EMVS.

Definition 11. An SCC $f : \mathcal{D}^n \to \mathcal{A}$ is an **extended median voter scheme (EMVS)** if there exists $\{a_S\}_{S \subseteq N}$ with $a_S \in \mathcal{I}$ for every $S \subseteq N$ satisfying the following two conditions

(i) $a_\emptyset = c_m$, $a_N = c_1$, and

(ii) $a_S \unlhd a_T$ for all $T \subseteq S$,

such that for all $R_N \in \mathcal{S}_\mathcal{I}$,

$$f(R_N) = \overline{\min}_{S \subseteq N}\{\overline{\max}_{i \in S}\{\tau(R_i), a_S\}\},$$

where $\overline{\min}$ and $\overline{\max}$ are the minimum and the maximum taken w.r.t. $\lhd$, respectively.[5]

Note that by Lemma 3, a tops-only, unanimous, and strategy-proof SCC on $\mathcal{S}_U$ selects an element of $\mathcal{I}$. Also, Theorem 1 in Rodríguez-Álvarez (2017) shows that any unanimous and strategy-proof SCC on $\mathcal{S}_\mathcal{I}$ is tops-only. These two together with the fact that for any preference $R \in \mathcal{S}_U \cup \mathcal{S}_\mathcal{I}$, $\tau(R) \in \mathcal{A}_1$ imply the following Corollary.

Corollary 1. *An SCC $f : \mathcal{S}_U^n \to \mathcal{A}$ is tops-only, unanimous, and strategy-proof if and only if there exists a unanimous and strategy-proof SCC $g : \mathcal{S}_\mathcal{I}^n \to \mathcal{I}$ such that $f(R_N) = g(R'_N)$ for all $R_N \in \mathcal{S}_U^n$ and all $R'_N \in \mathcal{S}_\mathcal{I}^n$ with $\tau(R_i) = \tau(R'_i)$ for all $i \in N$.*

[4] Rodríguez-Álvarez (2017) defines the preferences in a different manner, however, the preferences are single-peaked as defined here (see Lemma 1 in Rodríguez-Álvarez (2017)).

[5] Note that EMVS is well-defined on any domain $\mathcal{D}$ where $\tau(R) \in \mathcal{I}$ for all $R \in \mathcal{D}$. Therefore, it is well defined for all the domains we are considering in this paper.

In view of Corollary 1, Theorem 3 of Rodríguez-Álvarez (2017) and Theorem 2 of our paper together imply that any UAM rule is an EMVS and vice versa. Therefore, we have the following alternative characterization of tops-only, unanimous, and strategy-proof SCCs on $\mathcal{S}_U$.

Theorem 3. *An SCC $f : \mathcal{S}_U^n \to \mathcal{A}$ is tops-only, unanimous, and strategy-proof if and only if it is an EMVS.*

As the definitions of UAM rules and EMVS are structurally different, below we show directly that they are indeed equivalent.

Lemma 4. *An SCC is a UAM rule if and only if it is an EMVS.*

Proof of the only-if part: Let $f : \mathcal{D}^n \to \mathcal{A}$ be a UAM rule. We show that f is an EMVS. By definition, $f(R_N) = g(R_N) \cup h(R_N)$ for two adjacent min-max rules g and h. Let g have parameters denoted by β_S^g and h have parameters denoted by β_S^h. Since g and h are adjacent, we have $|[\beta_S^g, \beta_S^h]| \leq 2$ for all $S \subseteq N$; in particular, this implies that $\beta_S^g \cup \beta_S^h \in \mathcal{I}$ for all $S \subseteq N$. Define $a_S = \beta_S^g \cup \beta_S^h$ for all $S \subseteq N$. As $\{\beta_S^g\}_{S \subseteq N}$ and $\{\beta_S^h\}_{S \subseteq N}$ are parameters of two min-max rules, it follows that $a_\emptyset = c_m$, $a_N = c_1$, and $a_S \leq a_T$ for all $T \subseteq S$. Thus, $\{a_S\}_{S \subseteq N}$ satisfies all the conditions of the parameters of an EMVS. Additionally, for any set $S \subseteq N$, as $|[\beta_S^g, \beta_S^h]| \leq 2$, we must have either

$$\max_{i \in S}\{\tau(R_i), \beta_S^g\} = \max_{i \in S}\{\tau(R_i), \beta_S^h\} = \max_{i \in S}\{\tau(R_i), a_S\} \tag{1}$$

or

$$\beta_S^g = \max_{i \in S}\{\tau(R_i), \beta_S^g\} \neq \max_{i \in S}\{\tau(R_i), \beta_S^h\} = \beta_S^h, \text{ and } \max_{i \in S}\{\tau(R_i), a_S\} = a_S. \tag{2}$$

Consider the EMVS $\overline{f}$ generated by the parameters $\{a_S\}_{S \subseteq N}$. We prove that f is an EMVS by showing that f and $\overline{f}$ are identical. Let $R_N \in \mathcal{D}^n$. We first prove a claim showing that $\overline{f}(R_N) \trianglerighteq f(R_N)$.

Claim 2. $\overline{f}(R_N) \trianglerighteq f(R_N)$.

Proof of Claim 2: Assume for contradiction $\overline{f}(R_N) \vartriangleleft f(R_N)$ and let $V = \arg\min_{S \subseteq N}\{\max_{i \in S}\{\tau(R_i), a_S\}\}$. As either $\max_{i \in V}\{\tau(R_i), a_V\} \trianglerighteq \max_{i \in V}\{\tau(R_i), \beta_V^g\}$ or $\max_{i \in V}\{\tau(R_i), a_V\} \trianglerighteq \max_{i \in V}\{\tau(R_i), \beta_V^h\}$, it must be that either $\overline{f}(R_N) \trianglerighteq g(R_N)$ or $\overline{f}(R_N) \trianglerighteq h(R_N)$, a contradiction to $\overline{f}(R_N) \vartriangleleft f(R_N)$. Thus, $\overline{f}(R_N) \trianglerighteq f(R_N)$. □

We now distinguish two cases based on the values of $g(R_N)$ and $h(R_N)$.

Case 1: $g(R_N) \neq h(R_N)$.

Without loss of generality we may assume that $g(R_N) \prec h(R_N)$, while noting that $|[g(R_N), h(R_N)]| \leq 2$. Let $T = \arg\min_{S \subseteq N}\{\max_{i \in S}\{\tau(R_i), \beta^g_S\}\}$. Since $h(R_N) \succ g(R_N)$, it follows that $\max_{i \in T}\{\tau(R_i), \beta^g_T\} \prec \max_{i \in T}\{\tau(R_i), \beta^h_T\}$. This, together with $|[\beta^g_T, \beta^h_T]| \leq 2$, implies $h(R_N) = \max_{i \in T}\{\tau(R_i), \beta^h_T\}$. Hence, for T, (2) holds, implying

$$\beta^h_T = \max_{i \in T}\{\tau(R_i), \beta^h_T\} \rhd \max_{i \in T}\{\tau(R_i), a_T\} = a_T \rhd \max_{i \in T}\{\tau(R_i), \beta^g_T\} = \beta^g_T.$$

As $\overline{f}(R_N) \trianglerighteq f(R_N)$ (by Claim 2), thus, we have $\overline{f}(R_N) = \max_{i \in T}\{\tau(R_i), a_T\} = a_T = \beta^g_T \cup \beta^h_T = f(R_N)$.

Case 2: $g(R_N) = h(R_N)$.

Let $T_g = \arg\min_{S \subseteq N}\{\max_{i \in S}\{\tau(R_i), \beta^g_S\}\}$ and $T_h = \arg\min_{S \subseteq N}\{\max_{i \in S}\{\tau(R_i), \beta^h_S\}\}$. Suppose for T_g, (1) holds. Then, we have $g(R_N) = \max_{i \in T_g}\{\tau(R_i), \beta^g_{T_g}\} = \max_{i \in T_g}\{\tau(R_i), \beta^h_{T_g}\} = \max_{i \in T_g}\{\tau(R_i), a_{T_g}\}$ implying $f(R_N) \trianglerighteq \overline{f}(R_N)$. This, together with $\overline{f}(R_N) \trianglerighteq f(R_N)$, implies $\overline{f}(R_N) = f(R_N)$. Similar arguments apply if (1) holds for T_h. Thus, assume that (2) holds for both T_g and T_h. Therefore, by the assumption of the Case, we have

$$\beta^g_{T_g} = \max_{i \in T_g}\{\tau(R_i), \beta^g_{T_g}\} = \max_{i \in T_h}\{\tau(R_i), \beta^h_{T_h}\} = \beta^h_{T_h}. \tag{3}$$

Recall that by the definition of the β parameters of g and h, we have $\beta^g_{T_g} \succeq \beta^g_{T_g \cup T_h}$ and $\beta^h_{T_h} \succeq \beta^h_{T_g \cup T_h}$. This, together with $a_{T_g \cup T_h} = \beta^g_{T_g \cup T_h} \cup \beta^h_{T_g \cup T_h}$ and (3), implies $\beta^g_{T_g} = \beta^h_{T_h} \trianglerighteq a_{T_g \cup T_h}$. Also, as $\max_{i \in T_g}\{\tau(R_i), \beta^g_{T_g}\} = \beta^g_{T_g}$ (by (3)), it must be that $\max_{i \in T_g}\{\tau(R_i)\} \preceq \beta^g_{T_g}$. Similarly, $\max_{i \in T_h}\{\tau(R_i)\} \preceq \beta^h_{T_h}$. Combining the above observations, we have $\max_{i \in T_g \cup T_h}\{\tau(R_i), a_{T_g \cup T_h}\} \trianglelefteq \beta^g_{T_g} = \beta^h_{T_h}$. Hence, $\overline{f}(R_N) \trianglelefteq f(R_N)$ and by Claim 2, $\overline{f}(R_N) = f(R_N)$.

Since the two cases are exhaustive, this completes the proof that f is an EMVS. ■

Proof of the if part: Take an EMVS $\overline{f}$ with parameters $\{a_S\}_{S \subseteq N}$. Define $\beta^g_S = \min a_S$ and $\beta^h_S = \max a_S$ for all $S \subseteq N$. This means $\beta^g_S, \beta^h_S \in A$ for all $S \subseteq N$. Moreover, as $a_\emptyset = c_m$ and $a_N = c_1$, we have $\beta^g_\emptyset = \beta^h_\emptyset = c_m$ and $\beta^g_N = \beta^h_N = c_1$. Further, $\beta^g_S = \min a_S \trianglelefteq \min a_T = \beta^g_T$ implies $\beta^g_S \preceq \beta^g_T$ for all $T \subseteq S$. Similarly, we have $\beta^h_S \preceq \beta^h_T$ for all $T \subseteq S$. Thus, $\{\beta^g_S\}_{S \subseteq N}$ and $\{\beta^h_S\}_{S \subseteq N}$ satisfy all the conditions of the parameters of a min-max rule. Let g be the min-max rule with parameters $\{\beta^g_S\}_{S \subseteq N}$ and h the min-max rule with parameters $\{\beta^h_S\}_{S \subseteq N}$. Also, as $a_S \in \mathcal{I}$ for all $S \subseteq N$, we have $|[\beta^g_S, \beta^h_S]| \leq 2$ for all $S \subseteq N$, i.e., g and h are adjacent. Define a UAM rule $f : \mathcal{D}^n \to \mathcal{A}$ as $f(R_N) = g(R_N) \cup h(R_N)$ for all $R_N \in \mathcal{D}^n$. We claim that $\overline{f}$ and f are identical. Since $a_S = \beta_S^g \cup \beta_S^h$ for all $S \subseteq N$, it follows from the only-if part of this lemma that f and $\overline{f}$ are identical. Hence, every EMVS $\overline{f}$ is also a UAM rule. ■

4.2 Unanimous and Strategy-proof SCCs on $\mathcal{S}$

It is worth mentioning that, on both $\mathcal{S}_U$ and $\mathcal{S}_E$ domains, there are unanimous and strategy-proof SCCs that are not tops-only. Below we present an example of such an SCC. We think characterizing unanimous and strategy-proof SCC on both $\mathcal{S}_U$ and $\mathcal{S}_E$ is an interesting open problem that we leave for future research.

Example 4. Let $N = \{1,2\}$ and $A = \{a,b,c\}$ with $a \prec b \prec c$. Consider the following SCC f on $\mathcal{S}^2$.

$$f(R_1,R_2) = \begin{cases} b & \text{if } b \in \{\tau(R_1),\tau(R_2)\}, \\ a & \text{if } \tau(R_i) = a \text{ for all } i \in \{1,2\}, \\ c & \text{if } \tau(R_i) = c \text{ for all } i \in \{1,2\}, \\ b & \text{if } \{\tau(R_1),\tau(R_2)\} = \{a,c\} \text{ and } bP_i\{a,b,c\} \text{ for some } i \in \{1,2\}, \text{ and} \\ \{a,b,c\} & \text{if } \{\tau(R_1),\tau(R_2)\} = \{a,c\} \text{ and } \{a,b,c\}R_i b \text{ for all } i \in \{1,2\}. \end{cases}$$

It is straightforward to see that f is unanimous and non-tops-only. To see that f is strategy-proof, consider a profile (R_1,R_2). If $\tau(R_1) = b$, manipulation cannot happen: 1 gets their best possible outcome from reporting sincerely and 2 cannot change this outcome in any manner. The scenario is similar if $\tau(R_2) = b$. So we consider the only remaining case, where (WLOG, due to symmetry of f) $\tau(R_1) = a$ and $\tau(R_2) = c$. If $bP_1\{a,b,c\}$, then the outcome is b, and agent 1 can only change the outcome (possibly) to $\{a,b,c\}$ or c. But by our assumption $bP_1\{a,b,c\}$ and as her top is a, she prefers b over c (single-peakedness over the elements of $\mathcal{A}_1$). So, agent 1 cannot manipulate. If $\{a,b,c\}R_1 b$ then, if $bP_2\{a,b,c\}$ the outcome is b, otherwise it is $\{a,b,c\}$. If the outcome is b then agent 1 can change it only to c and that she does not prefer over b (single-peakedness over the elements of $\mathcal{A}_1$). Finally, if the outcome is $\{a,b,c\}$ then she can change it to b or c. But as she prefers $\{a,b,c\}$ over b, she can not manipulate in this situation. A similar reasoning also holds for 2. This shows that f is strategy-proof. □

5. APPENDIX

5.1 PROOF OF FACT 1

We provide a proof for Fact 1, assuming the Theorem (Moulin (1980), Weymark (2011)):

Proof. We introduce some terminologies to ease the presentation of the proof. For $Q \in \mathcal{S}|_{\mathcal{A}_1}$, we use the notation R to denote an "extension" of Q to a preference in $\mathcal{S}$, that is, $R \in \mathcal{S}$ is such that if we restrict R to the alternatives in $\mathcal{A}_1$, we obtain Q, more formally, $R|_{\mathcal{A}_1} = Q$. Similarly, for $R \in \mathcal{S}$, by Q we denote the restriction of R to the alternatives in $\mathcal{A}_1$. Moreover, $R_N|_{\mathcal{A}_1} = Q_N$.

Let $f : \mathcal{S}^n \to A$ be a unanimous, strategy-proof, and tops-only SCF. Define an SCF $g : (\mathcal{S}|_{\mathcal{A}_1})^n \to A$ given by $g(Q_N) = f(R_N)$ for all $Q_N \in (\mathcal{S}|_{\mathcal{A}_1})^n$. Since $\tau(Q) = \tau(R)$ for all $Q \in \mathcal{S}$ and f is tops-only, g is well-defined and tops-only. Further, as f is unanimous and strategy-proof, g must also be unanimous and strategy-proof. Therefore, by the Theorem (Moulin (1980), Weymark (2011)), g is a min-max rule. So, for every $S \subseteq N$, there exists $\beta^g_S \in A$ with $\beta^g_\emptyset = c_m$, $\beta^g_N = c_1$, and $\beta^g_T \preceq \beta^g_S$ for $S \subseteq T$ such that for all $Q_N \in (\mathcal{S}|_{\mathcal{A}_1})^n$,

$$g(Q_N) = \min_{S \subseteq N}[\max_{i \in S}\{\tau(Q_i), \beta^g_S\}].$$

But $g(Q_N) = f(R_N)$ and $\tau(Q_i) = \tau(R_i)$. Hence for all $R_N \in \mathcal{S}^n$,

$$f(R_N) = \min_{S \subseteq N}[\max_{i \in S}\{\tau(R_i), \beta^g_S\}].$$

So, $f : \mathcal{S}^n \to A$ is a min-max rule.

For the converse, let $f : \mathcal{S}^n \to A$ be a min-max rule. Thus, f is unanimous and tops-only by definition. We show that f is strategy-proof as well. Define a function $g : (\mathcal{S}|_{\mathcal{A}_1})^n \to A$ given by $g(Q_N) = f(R_N)$. Note that g is well-defined as $\tau(Q) = \tau(R)$ for all $Q \in \mathcal{S}|_{\mathcal{A}_1}$ and f is tops-only. Since f is a min-max rule, for every $S \subseteq N$ there exists $\beta^f_S \in A$ with $\beta^f_\emptyset = c_m$, $\beta^f_N = c_1$, and $\beta^f_T \preceq \beta^f_S$ for $S \subseteq T$ such that for all $R_N \in \mathcal{S}^n$

$$f(R_N) = \min_{S \subseteq N}[\max_{i \in S}\{\tau(R_i), \beta^f_S\}].$$

But $g(Q_N) = f(R_N)$ and $\tau(Q_i) = \tau(R_i)$. Hence for all $Q_N \in (\mathcal{S}|_{\mathcal{A}_1})^n$,

$$g(Q_N) = \min_{S \subseteq N}[\max_{i \in S}\{\tau(Q_i), \beta^f_S\}].$$

So, g is a min-max rule. From the Theorem (Moulin (1980), Weymark (2011)), g is strategy-proof. If the strategy-proofness of f were contradicted at some profile, the strategy-proofness of g would be contradicted at the corresponding restricted profile, so f must be strategy-proof. ■

5.2 An important lemma

The following lemma guarantees the existence of different preferences considered in the proofs.

Lemma 5. *The following existence statements are true:*

(i) For all $x, x_i \in A$ with $x \prec x_i$, there exists $R_1 \in \mathcal{S}_E$ such that $\tau(R_1) = x_i$ and for all $X, Y \subseteq A$, YP_1X whenever $[c_1, x] \cap Y = \emptyset$ and $[c_1, x] \cap X \neq \emptyset$.

(ii) For any $S \subseteq A$ with $|S| > 2$, if $s_1 = \min S$ and $s_2 = \max S$,

- *(a) there exists $R_2 \in \mathcal{S}_E$ such that $\tau(R_2) = s_1 + 1$ and $SP_2\{s_1\}$,*
- *(b) there exists $R_3 \in \mathcal{S}_E$ such that $\tau(R_3) = s_2$ and $\{s_1 + 1\}P_3S$, and*
- *(c) there exists $R_4 \in \mathcal{S}_E$ such that $\tau(R_4) = s_1 + 1$ and $SP_4\{s_1, s_1 + 1\}$.*

(iii) For any $x, a_i \in A$ and $z \in \{a_i, a_i + 1\}$, such that $x \succ a_i$, there exists $R_5 \in \mathcal{S}_E$ such that $\tau(R_5) = z$ and for all $X, Y \subseteq A$, XP_5Y whenever $[x, c_m] \cap X = \emptyset$ and $[x, c_m] \cap Y \neq \emptyset$.

(iv) For any $y, x, a_i \in A$ and $z \in \{a_i, a_i + 1\}$, such that $y \succ x \succ a_i$, there exists $R_6 \in \mathcal{S}_E$ such that $\tau(R_6) = z$ and for all $X, Y \subseteq A$, XP_6Y whenever $[c_1, x] \cap X \neq \emptyset$ and $[c_1, x] \cap Y = \emptyset$.

(v) For any $b \in A, Y_j, Y_{j-1} \subseteq A$ such that $b = \max Y_j \prec \max Y_{j-1}$ and $b \notin Y_{j-1}$, there exists $R_7 \in \mathcal{S}_U$ such that for all $k, l \in A$ with $k \prec b \prec l$, bP_7lP_7k, and $Y_{j-1}P_7Y_j$.

Proof. Below we prove the above statements one by one. We use the following terminology in the proof. For a preference $R_j \in \mathcal{S}$, we denote v_j and λ_j for a corresponding utility function and a corresponding assessment, respectively.

(i) Let R_1 be such that $v_1(y) < -100 \cdot |A|$ for $y \preceq x$, and it takes values between 0 and 1 for $y \succ x$, peaking at x_i, where $x_i \succ x$ by assumption. This can be done by setting $v_1(x + 1) = 0, v_1(x_i) = 1, v_1(c_m) = 0.1$, and interpolating for all candidates in between. Now, for any X such that $[c_1, x] \cap X \neq \emptyset$, $v_1^E(X) < -99 \cdot |A|$. For any Y such that $[c_1, x] \cap Y = \emptyset$, $v_1^E(Y) \geq 0$. Hence, YP_1X.

(ii) (a) Consider R_2 such that $v_2(s_1) = 0.1$. Set $v_2(x) > 10$ for all $x \succ s_1$ such that v_2 attains its maximum at $s_1 + 1$, for example by setting $v_2(s_1 + 1) = 12, v_2(c_m) = 11$, and interpolating for all candidates between $s_1 + 1$ and c_m. Then clearly $v_2^E(S) > v_2^E(\{s_1\})$.

(b) Consider R_3 such that $v_3(s_1) = -100 \cdot |A| - 1$. Set $0 \leq v_2(x) \leq 1$ for all $x \succ s_1$ such that v_3 attains its maximum at s_2, for example by setting $v_3(s_1+1) = 0, v_3(s_2) = 1$, and interpolating for all candidates in between. Then $v_3^E(S) < -99 \cdot |A| < 0 \leq v_3^E(\{s_1+1\})$.

(c) Consider R_4 such that $v_4(s_1) = 0, 10 < v_4(x) < 13$ for all $x \succ s_1$ such that v_4 peaks at s_1+1, by setting $v_4(s_1+1) = 12, v_4(c_m) = 11$, and interpolating for all candidates in between s_1+1 and c_m. Then $v_4^E(\{s_1, s_1+1\}) = 6 < \frac{20}{3} \leq v_4^E(S)$, with the last inequality following from the fact that $v_4^E(S)$ is the average of 0 and at least 2 other real numbers, each of which is greater than 10.

The other functions can also be constructed with similar techniques, so a sketch of the idea is given in place of the explicit construction:

(iii) Consider R_5 such that $v_5(a) < -100 \cdot |A|$ for all $a \succeq x$ and $v_5(a) > 0$ for all $a \prec x$; such that $v_5(a_i)$ or $v_5(a_i+1)$ is the peak as required.

(iv) Consider R_6 such that $v_6(a) > 100 \cdot |A|$ for all $a \prec x+1$ and $v_6(a) < 0$ for all $a \succeq x+1$; such that $v_6(a_i)$ or $v_6(a_i+1)$ is the peak as required.

(v) Consider R_7 such that $v_7(b) = 100, v_7(l) > 99$ for all $l \succ b, 0 < v_7(k) < 0.1$ for all $k \prec b$, and $\lambda_6(l) > 100 \cdot |A| \cdot \lambda_6(k) > 10000 \cdot |A| \cdot \lambda_6(b)$ for all k, l such that $k \prec b \prec l$.

This completes the proof of the lemma. ■

5.3 Proof of Lemma 2

Proof. Since f is tops-only, we can recast it as a function $f : A^n \to \mathcal{A}$. We first prove the result for two voters. Assume for contradiction the lemma does not hold, that is, there exist $x_1, x_2 \in A$ such that $|f(x_1, x_2)| > 2$. WLOG assume that $x_1 \prec x_2$. Define $S := f(x_1, x_2)$, $s_1 := \min S$, and $s_2 := \max S$. From strategy-proofness of f, $f(x_1, x_2) = S = f(s_1, s_2)$. We look at $f(s_1, s_1+1)$. Since $|S| > 2$, $s_1 + 1 \neq s_2$. By Proposition 1, $f(s_1, s_1+1) = \{\{s_1\}, \{s_1+1\}, \{s_1, s_1+1\}\}$. First assume that $f(s_1, s_1+1) = \{s_1\}$. If agent 1 votes for s_1, consider the case where agent 2 has a preference R_1 such that $\tau(R_1) = s_1 + 1$ and $SP_1\{s_1\}$ (see (ii)-(a) of Lemma 5 in Appendix 5.2). Agent 2 can manipulate by voting for s_2, giving rise to a contradiction. Now assume $f(s_1, s_1+1) = \{s_1+1\}$. If agent 1 votes for s_1, consider the case where agent 2 has a preference R_2 such that $\tau(R_2) = s_2$ and $\{s_1+1\}P_2S$ (see (ii)-(b) of Lemma 5 in Appendix 5.2). Agent 2 can manipulate by voting for s_1+1, giving rise to a contradiction. Finally, assume that $f(s_1, s_1+1) = \{s_1, s_1+1\}$. If agent 1 votes for s_1, consider the case where agent 2 has a preference

R_3 such that $\tau(R_3) = s_1 + 1$ and $SP_3\{s_1, s_1 + 1\}$ (see (ii)-(c) of Lemma 5 in Appendix 5.2). Agent 2 can manipulate by voting for s_2, giving rise to a contradiction. Thus, we have shown that the lemma holds for $n = 2$. Next, we introduce an induction hypothesis.

Induction Hypothesis: Given $n \geq 3$, all tops-only, unanimous, and strategy-proof SCC $g : \mathcal{S}_E^{n-1} \to \mathcal{A}$ has sets of size at most 2 in the range.

We show that all tops-only, unanimous, and strategy-proof SCC $f : \mathcal{S}_E^n \to \mathcal{A}$ has the same property. Consider an SCC $f : \mathcal{S}_E^n \to \mathcal{A}$ that is tops-only, unanimous, and strategy-proof. As f are tops-only, we rewrite f as a function with domain A^n. Define a tops-only SCC g on $\mathcal{S}_E^{n-1}$ in the following way:

$$g(x_1, x_3, \ldots, x_n) = f(x_1, x_1, x_3 \ldots, x_n) \text{ for all } (x_1, x_3, \ldots, x_n) \in A^{n-1}.$$

Note that g is a tops-only SCC with voters $1, 3, \ldots, n$. Further, as f is unanimous, g is also unanimous. We show that g is strategy-proof. Clearly individuals $3, \ldots, n$ cannot manipulate g, as otherwise they would be able to manipulate f. Suppose then 1 could manipulate g. Then we have:

$$\begin{aligned} J &:= f(a_1, a_1, a_3, \ldots, a_n) = g(a_1, a_3, \ldots, a_n) \\ K &:= f(b_1, a_1, a_3, \ldots, a_n) \\ L &:= f(b_1, b_1, a_3, \ldots, a_n) = g(b_1, a_3, \ldots, a_n) \end{aligned}$$

and a preference ordering R_1 for voter 1 with $\tau(R_1) = b_1$ such that JP_1L (in particular, $\{b_1\}P_1\{a_1\}$). We look at f when voters 1 and 2 have the same ordering R_1. Since agent 1 cannot manipulate f, KR_1J. Similarly, as agent 2 cannot manipulate f, LR_1K. Combining the two observations, we have LR_1J, a contradiction to JP_1L. Thus, g is strategy-proof.[6] Hence, by the induction hypothesis, the range of g has sets of size at most 2.

Now we show that the range of f has sets of size at most 2. Suppose the contrary; let $f(a_1, \ldots, a_n)$ have size more than 2. If any two of $a_1, \ldots, a_n$ are identical — WLOG say $a_1 = a_2$ — then we define $g(x_1, x_3, \ldots, x_n) = f(x_1, x_1, x_3, \ldots, x_n)$. As g is unanimous and strategy-proof as seen above, by the induction hypothesis, $|g(a_1, a_3, \ldots, a_n)| \leq 2$. Thus, $|f(a_1, a_1, a_3, \ldots, a_n)| \leq 2$. So, assume that no two of $a_1, \ldots, a_n$ are identical. WLOG let $a_1 \prec a_2 \prec \ldots \prec a_n$. Let $S = f(a_1, a_2, a_3, \ldots, a_n)$ and $T = f(a_2, a_2, a_3, \ldots, a_n)$. Note that $|T| \leq 2$. If $S \cap [a_1, a_2 - 1] = \emptyset$, then 1 can manipulate by choosing their preferred outcome among $f(a_1, \ldots, a_n)$ and $f(a_2, a_2, a_3, \ldots,$

[6]We have included the proof of strategy-proofness of g for completeness. It can also be found in Sen (2001).

a_n) regardless of whether they prefer a_1 or a_2, breaking strategy-proofness. So there exists $x \in S \cap [a_1, a_2 - 1]$. Similarly there exists $y \in S \cap [a_{n-1} + 1, a_n]$. Let y be the maximum such candidate. We note that $\max S = \max T$, since otherwise, if $\max S \succ \max T$, 1 can manipulate at $(a_1, a_2, \ldots, a_n)$ by voting a_2 when they have a utility function that goes arbitrarily low for all candidates a with $a \succeq \max S$ (and similarly if $\max T \succ \max S$, 1 can manipulate at $(a_2, a_2, \ldots, a_n)$). Now, let 1 have a utility v_i that peaks at a_2 such that $v_i(a_2) - v_i(c) < 0.01$ for all candidates $c \prec y$, and $v_i(a_2) - v_i(d) > 100$ for all candidates d such that $d \succeq y$. Then 1 can manipulate at $(a_2, a_2, a_3, \ldots, a_n)$ by voting a_1, since the outcome set S would have a larger number of candidates with high utility than the outcome set T, and hence would be preferred by 1. ■

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